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On mass-conserving solutions of the discrete coagulation equation

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Abstract. We present the explicit pre-gelation solution of the discrete coagulation equation for the general bilinear kernel and arbitrary initial conditions. For the pure addition kernel, we prove that gelation never takes place even if the second moment of the initial particle distribution is infinite.

1. Introduction

The number distribution of an assembly of rapidly coagulating particles, each of integer mass, is governed by the infinite system of equations

$$\dot{c}_i \equiv \frac{\mathrm{d}c_i}{\mathrm{d}t} = \frac{1}{2} \sum_{j+k=i} K_{jk} c_j c_k - c_i \sum_{j=1}^{\infty} K_{ij} c_j \qquad i = 1, 2, 3, \ldots \tag{1}$$

where $c_i(t)$ is the number of particles of mass i at time t, and K_{ij} is the coagulation kernel (von Smoluchowski 1916). Equations (1) must be supplemented by a given set of non-negative initial values $c_i(0)$ representing the initial size distribution. In the case where all the particles are initially of unit mass (the monodisperse initial conditions), system (1) was solved by Smoluchowski for the constant kernel $K_{ij} = 1$ and by McLeod (1962) for the product kernel $K_{ij} = ij$. In the latter case, McLeod found that no solution exists which conserves the total mass for all time. Indeed, very little is known about general properties of the solutions of (1) for unbounded kernels (Drake 1972). White (1980) has shown that, for the addition kernel $K_{ij} = i + j$, the nth moment of the distribution, $S_n \equiv \Sigma_i i^n c_i$, remains finite if it is initially finite, for $n \geqslant 2$. It follows that for this kernel mass is conserved if the second moment is initially finite.

There has been much interest in the bilinear kernel $K_{ij} = A + B(i+j) + Cij$ (where A, B, C are non-negative constants), mainly because this appears to be the only form which offers any hope of analytical solution, although it does have physical significance (Cohen and Benedek 1982). Much work has concentrated on monodisperse initial conditions and this case has now essentially been solved (van Dongen and Ernst 1984). When $C > 0$, it is easy to show directly from (1) that (provided we can interchange $\mathrm{d}/\mathrm{d}t$ with summations) the total number of particles becomes negative at a finite time, t_0, if it is assumed that the total mass is conserved. Also, if S_3 is assumed finite, then the second and higher moments tend to infinity at a finite time, t_m. As a result, it was long believed that the product kernel was physically unrealistic, but it is now known that for monodisperse initial conditions a solution of (1) exists beyond $t = t_m$, with

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the total mass decreasing for $t > t_m$ (Ziff and Stell 1980). It must be emphasised that the equation for $\dot{S}_n$ derived directly from (1) (Drake 1972) is dependent on the assumption that S_{n+1} exists, and is therefore not *a priori* valid. Any attempt to derive exact solutions of (1) which fails to take account of this point (e.g. Spouge 1983, Lu Binglin 1987) is clearly inadequate. It is straightforward to show that $\dot{S}_1 \leqslant 0$ (at least while $c_i \geqslant 0$ for all i), and in polymer science a decreasing total mass is interpreted physically as indicating the creation of a gel, coexisting with matter still in the sol phase (Leyvraz and Tschudi 1982, Ziff *et al* 1983, van Dongen and Ernst 1984). The gel time, t_g, is defined as the point at which the total mass ceases to be constant. Evidently, if $S_2 \to \infty$ as $t \to t_m$, it is necessarily the case that $t_m \leqslant t_g$. However, it is not obvious that $t_m = t_g$ as has sometimes been assumed (van Dongen and Ernst 1983, Ziff *et al* 1983). The present authors have shown rigorously that the mass S_1 is conserved and that $S_2(\frac{1}{2}t) = S_1^2/S_0(t)$ for $t < t_m$ in the monodisperse case when $C > 0$. The existence of the post-gel solution depends upon continuity through the gel point, which we have not established. Subject to this, we have $t_g = t_m = \frac{1}{2}t_0$.

In this paper, we determine the form that a mass-conserving solution of (1) must take if it exists, for arbitrary A, B, C and initial conditions (subject only to the existence of the initial total mass). For the addition kernel we prove rigorously that gelation never occurs (note that this is immediate for the constant kernel). The 'usual' moment equation for $S_n(t)$ remains valid provided $S_n(0)$ is finite (even if $S_{n+1}(0)$ is infinite), while if $S_n(0)$ is infinite, then $S_n(t)$ remains infinite for all t. In particular ($n = 1$), this demonstrates that gelation is not necessarily associated with an infinite second moment.

2. The general solution

Let $p(t) = \Sigma_{i=1}^{\infty} c_i(t)$ and $m(t) = \Sigma_i ic_i(t)$ be the total number of particles and their total mass respectively, and set $M = m(0)$. If there is a mass-conserving solution to (1) then we may add (1) for all i and deduce that p satisfies the differential equation $\mathrm{d}p/\mathrm{d}t = -\frac{1}{2}(Ap^2 + 2BpM + CM^2)$. We accordingly define $P(t)$ by $P(0) = p(0)$ and $\mathrm{d}P/\mathrm{d}t = -\frac{1}{2}(AP^2 + 2BMP + CM^2)$. Now consider the following system of equations:

$$\dot{X}_i + [(AP + BM) + (BP + CM)i]X_i = \tfrac{1}{2} \sum_{j+k=i} K_{jk}X_jX_k \tag{2}$$

subject to the initial conditions $X_i(0) = c_i(0)$ for all i. In theorem 2 we obtain explicit solutions to (2), and show in particular that $X_i \geqslant 0$ for all $t \geqslant 0$. Before doing so, however, we establish the connection between (1) and (2).

Theorem 1. The following are equivalent for any $0 < T \leqslant \infty$.

(a) System (1) has a solution $\{c_i(t)\}$ with $c_i \geqslant 0$ and $m(t) = M$ for all $t \in [0, T)$.

(b) $X_i = c_i$ and $\Sigma_{i=1}^{\infty} iX_i(t)$ is continuous for all $t \in [0, T)$.

(c) $\Sigma_{i=1}^{\infty} iX_i(t) = M$ for all $t \in [0, T)$.

Moreover if $C = 0$, then the above are implied by

(d) $\Sigma_{i=1}^{\infty} i^2X_i(t)$ is continuous for all $t \in [0, T)$.

Proof. In what follows we write $\pi = \Sigma_i X_i$ and $\mu = \Sigma_i iX_i$. Note that, if μ is continuous in t, then the positivity of the terms iX_i (proved in theorem 2) and a well known theorem of Dini (Bromwich 1926, § 49.2) imply that μ is uniformly convergent.

Summing (2) over all i it follows that $\Sigma \dot{X}_i$ is uniformly convergent. Thus $\dot{\pi} = \Sigma \dot{X}_i$, whence

$$\dot{\pi} + (AP + BM)\pi + (BP + CM)\mu = \tfrac{1}{2}\sum_{i=1}^{\infty}\sum_{j+k=i} K_{jk}X_jX_k.$$

In terms of $\xi = \pi - P$ and $\eta = \mu - M$ this reduces to

$$\dot{\xi} = \tfrac{1}{2}(A\xi^2 + 2B\xi\eta + C\eta^2). \tag{3}$$

$(b)\to(a)$. Since $X_i = c_i$ we have $\mu = m$ and $\pi = p$, and it follows from (1) and (2) that $AP + BM = Ap + Bm$ and $BP + CM = Bp + Cm$. In terms of $\xi = \pi - P = p - P$ and $\eta = m - M$ these reduce to $A\xi + B\eta = 0 = B\xi + C\eta$. But then (3) implies that $2\dot{\xi} = \xi(A\xi + B\eta) + \eta(B\xi + C\eta) = 0$, and so ξ is identically zero since $\xi(0) = 0$. Thus $B\eta = C\eta = 0$, and hence $\eta = 0$ identically except possibly when $B = C = 0$. In this rather trivial case (the constant kernel) we may multiply (1) by i and sum to give $m = M$. Thus in all cases $m = M$ and $p = P$. Therefore (2) reduces to (1) and we have (a).

$(c)\to(b)$. Here by assumption $\eta = \mu - M = 0$ for all $t\in[0, T)$, and so (3) gives $\dot{\xi} = \tfrac{1}{2}A\xi^2$. In view of $\xi(0) = 0$ this implies that $\xi(t) = 0$ for all $t \in [0, T)$. We now know that $M = \Sigma\, iX_i$ and $P = \Sigma X_i$. Substituting these into (2), the system of equations reduces to (1), so $X_i = c_i$ since $X_i(0) = c_i(0)$.

$(a)\to(c)$. If $m(t) = M$ for all $t \in (0, T)$ then the same argument as at the beginning of this proof, using (1) instead of (2), implies that $\dot{p} = \Sigma\, \dot{c}_i$. Summing (1) over all i we find that $\dot{p} = -\tfrac{1}{2}(Ap^2 + 2BpM + CM^2)$. Hence p and P satisfy the same differential equation, and as $P(0) = p(0)$ we must have $P = p$ identically. Thus (1) and (2) are identical, so $X_i = c_i$ for all i, and in particular $\mu = m = M$.

If $C = 0$, $(d)\to(c)$. Denote $\Sigma\, i^2X_i$ by σ. Multiplying (2) by i and summing and using the continuity of σ, we find

$$\dot{\mu} = A\xi\mu + B(\mu\eta + \sigma\xi) + C\sigma\eta.$$

If $C = 0$, then $\xi = 0$ by (3). Hence $\dot{\eta} = B\eta(\eta + M)$ and it follows that $\eta = 0$, i.e. $\mu = M$.

We proceed to the solution of (2). It is convenient to work in terms of the variable $u = P/M$. It is immediate that u satisfies the equation $\mathrm{d}u/\mathrm{d}t = -\tfrac{1}{2}M(Au^2 + 2Bu + C)$. The functional form of u is given in the appendix. Also, in what follows we adopt the convention that the subscript zero denotes the value of a variable at time $t = 0$. We then have the following.

Theorem 2. The system of equation (2) has the solution

$$X_i = \tfrac{1}{2}Mq(u)Q_i(u)\alpha(u)^{-i}$$

for all i, where $q(u) = Au^2 + 2Bu + C$, the function α satisfies $\alpha(u_0) = 1$ and $\mathrm{d}\ln\alpha/\mathrm{d}u = -2(Bu + C)/q(u)$, and the Q_i are polynomials of degree at most $i - 1$ in u. More precisely, $Q_i(u) = \Sigma_{j=0}^{i-1} a_{ij}(u_0 - u)^j$ where $a_{i0} = 2c_i(0)/Mq(u_0)$ and for $j \geq 1$

$$a_{ij} = (2j)^{-1}\sum_{s=1}^{i-1}\sum_{k=0}^{j-1} K_{s,i-s}a_{s,k}a_{i-s,j-k-1}. \tag{4}$$

Proof. Clearly (2) has the integrating factor ρ_i given by $\dot{\rho}_i/\rho_i = (AP + BM) + (BP + CM)i$. It follows that $\rho_i = \alpha^i\beta$, where α is as above and β satisfies $\dot{\beta}/\beta = M(Au + B)$, $\beta(u_0) = 1$. Consequently $\beta(u) = q(u_0)/q(u)$. Since $\rho_i = \beta^{-1}\rho_j\rho_k$

when $i=j+k$, equation (2) can be written in the form $\mathrm{d}(\rho_i X_i)/\mathrm{d}t = (2\beta)^{-1}\sum_{j+k=i} K_{jk}(\rho_j X_j)(\rho_k X_k)$. Now set $Q_i(u)=2\rho_i X_i/Mq(u_0)$. Using the formulae for $\dot{u}$ and β we obtain

$$\mathrm{d}Q_i/\mathrm{d}u = -\tfrac{1}{2}\sum_{j+k=i} K_{jk}Q_jQ_k. \tag{5}$$

Clearly $Q_i(u_0)=2c_i(0)/Mq(u_0)$ for all i. For $i=1$ we have $\mathrm{d}Q_1/\mathrm{d}u=0$, and so $Q_1(u)\equiv 2c_1(0)/Mq(u_0)$. Assume inductively that $Q_s(u)$ is a polynomial of degree less than s for all $s<i$. Then the right-hand side of (5) is a polynomial of degree at most $j-1+k-1=i-2$, and so Q_i has degree at most $i-1$. Write $Q_i(u)=\sum_{j=0}^{i-1} a_{ij}(u_0-u)^j$, so $Q_i^{(j)}(u_0)=(-1)^j j!\,a_{ij}$. Differentiating (5) $j-1$ times and setting $u=u_0$ we find

$$\begin{aligned}(-1)^j j!\,a_{ij} &= -\tfrac{1}{2}\sum_{s=1}^{i-1}\sum_{k=0}^{j-1} K_{s,i-s}\binom{j-1}{k} Q_s^{(k)}(u_0)Q_{i-s}^{(j-1-k)}(u_0)\\ &= -\tfrac{1}{2}\sum_s\sum_k K_{s,i-s}\binom{j-1}{k}(-1)^k k!\,a_{s,k}(-1)^{j-k-1}(j-k-1)!\,a_{i-s,j-k-1}\end{aligned}$$

where we have used the form $-2^{-1}\sum_{s=1}^{i-1} K_{s,i-s}Q_sQ_{i-s}$ for the sum in (5). The result follows.

Remark. It is easy to see that $\alpha(u)$ is equal to $\exp[2(u_0-u)]$ when $A=B=0$, to $\exp(u_0-u)[(u_0+C/2B)/(u+C/2B)]^{C/2B}$ when $B>0$ and $A=0$, and to $[q(u_0)/q(u)]^{B/A}\exp[(AC-B^2)Mt/A]$ otherwise. In particular, $\alpha\geqslant 0$, and so $X_i\geqslant 0$, for all u. This establishes the property of the X_i used in the proof of theorem 1.

3. Moments and initial conditions

We proceed to derive more convenient forms for the moments $S_n(u)=\sum_{i=1}^{\infty} i^n X_i(u)$ (it would be pedantic to use a different symbol from S here, since we cannot derive results on the moments of the c_i before establishing that $c_i=X_i$ for all i). Define $\psi(z)=\sum_{i=1}^{\infty} a_{i0}z^i = 2(Mq_0)^{-1}\sum_i c_i(0)z^i$, where z is a complex variable. Since $\psi(1)$ exists by assumption, ψ is analytic in the disc $|z|<1$. Also, if $|z|\leqslant 1$, then $|\sum_{i=1}^{n} a_{i0}z^i|\leqslant \sum_1^n a_{i0}|z|^i \leqslant (\sum_1^n a_{i0})|z|\leqslant \psi(1)|z|$. Letting $n\to\infty$ we obtain

$$|\psi(z)|\leqslant \psi(1)|z|.$$

We now have the following.

Theorem 3. For $n, s\geqslant 0$ define $L_{n,s}(u)=\sum_{i=1}^{\infty} i^n a_{is}\alpha(u)^{-i}$ for all $u\leqslant u_0$. Then:

(i) for all A, B, C we have $S_n(u)=\tfrac{1}{2}Mq(u)\sum_{s=0}^{\infty} L_{n,s}(u)(u_0-u)^s$;

(ii) if B and C are not both zero then $L_{n,s}(u)$ is finite for all $u<u_0$, and we have

$$L_{n+1,s}(u)=\frac{q(u)}{2(Bu+C)}\frac{\mathrm{d}}{\mathrm{d}u}(L_{n,s})$$

$$L_{n,s}=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}(AL_{r,k}L_{n-r,s-k-1}+2BL_{r+1,k}L_{n-r,s-k-1}+CL_{r+1,k}L_{n-r+1,s-k-1})$$

for all $n, s\geqslant 0$ and all $u<u_0$.

(i) is to be interpreted as meaning that the two sides are either both finite and equal, or both infinite, for any given value of u.

Proof. (i) By theorem 2 we can write

$$S_n(u)=\tfrac{1}{2}Mq(u)\sum_{i=1}^{\infty}\sum_{s=0}^{\infty} i^n a_{is}(u_0-u)^s\alpha(u)^{-i}.$$

Thus (i) is a consequence of Pringsheim's theorem on double series of positive terms (Bromwich 1926, §31).

(ii) Define a sequence of functions ϕ_s by

$$\phi_0(z)=\psi(e^z)$$

$$\phi_s=\frac{1}{2s}\sum_{k=0}^{s-1}(A\phi_k\phi_{s-k-1}+2B\phi'_k\phi_{s-k-1}+C\phi'_k\phi'_{s-k-1}) \qquad s\geqslant 1$$

where differentiation is with respect to z. Since ψ is analytic in the open unit disc, ϕ_0 (and hence all the ϕ_s) are analytic for $\mathrm{Re}(z)<0$. For $u<u_0$ let $v=\ln\alpha(u)$. Then $v>0$ since B and C are not both zero (cf theorem 2), and hence all the derivatives $\phi_s^{(n)}(-v)$ exist. We show that $L_{n,s}(u)=\phi_s^{(n)}(-v)$ for all $n,s\geqslant 0$. To begin with, $\phi_0(z)=\Sigma_i a_{i0}e^{iz}$, so $\phi_0^{(n)}(-v)=\Sigma_i i^n a_{i0}e^{-iv}=\Sigma_i i^n a_{i0}\alpha(u)^{-i}=L_{n,0}$ for all n. Assume inductively that $L_{n,j}(u)=\phi_j^{(n)}(-v)$ for all $n\geqslant 0$ and all $j<s$.

Differentiating the defining equation for ϕ_s n times we obtain

$$\phi_s^{(n)}=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}(A\phi_k^{(r)}\phi_{s-k-1}^{(n-r)}+2B\phi_k^{(r+1)}\phi_{s-k-1}^{(n-r)}+C\phi_k^{(r+1)}\phi_{s-k-1}^{(n-r+1)}).$$

Let $z=-v$ and use the inductive hypothesis to get

$$\phi_s^{(n)}(-v)=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}(AL_{r,k}L_{n-r,s-k-1}+2BL_{r+1,k}L_{n-r,s-k-1}+CL_{r+1,k}L_{n-r+1,s-k-1}).$$

Now by (3) we have

$$a_{is}=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{e+f=i}K_{e,f}a_{e,k}a_{f,s-k-1}$$

and so

$$\begin{aligned}i^n a_{is}&=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{e+f=i}(e+f)^n[A+B(e+f)+Cef]a_{e,k}a_{f,s-k-1}\\&=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}\sum_{e+f=i}(Ae^rf^{n-r}+2Be^{r+1}f^{n-r}+Ce^{r+1}f^{n-r+1})a_{e,k}a_{f,s-k-1}.\end{aligned}$$

Now note, for example, that the product $L_{r,k}L_{n-r,s-k-1}$ in the expression for $\phi_s^{(n)}(-v)$ is equal to

$$\left(\sum_{e=1}^{\infty}e^r a_{e,k}\alpha^{-e}\right)\left(\sum_{f=1}^{\infty}f^{n-r}a_{f,s-k-1}\alpha^{-f}\right)=\sum_{i=1}^{\infty}\left(\sum_{e+f=i}e^rf^{n-r}a_{e,k}a_{f,s-k-1}\right)\alpha^{-i}.$$

Writing down similar expressions for the other products in $\phi_s^{(n)}(-v)$ and using the above formula for $i^n a_{is}$ to simplify the result it is easy to see that

$$\phi_s^{(n)}(-v)=\sum_{i=1}^{\infty}i^n a_{is}\alpha^{-i}=L_{n,s}(u) \qquad \text{for all } n.$$

We have therefore established the finiteness of the $L_{n,s}$ and the second formula in (ii). The first formula in (ii) follows from $L_{n,s}(u)=\phi_s^{(n)}(-v)$ and the fact that $\mathrm{d}v/\mathrm{d}u=\mathrm{d}(\ln\alpha)/\mathrm{d}u=-2(Bu+C)/q(u)$.

The constant kernel case ($B = C = 0$) is not covered by theorem 3. It is, however, easy to deal with directly.

Theorem 4. If $A=1$, $B=C=0$, then $\pi=P$, $\mu=M$ and $c_i=X_i$ for all $t\geqslant 0$. Furthermore, if S_n is initially finite, it is given for all t by the formula

$$S_n(u)=\tfrac{1}{2}Mu^2\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\left(\frac{1}{\phi(z)}-\frac{u_0-u}{2}\right)^{-1}\right]_{z=0}.$$

Proof. We first note that $q(u)=u^2$ and $\alpha=1$. Assume that $S_n(u_0)<\infty$ for $n\leqslant m$, for some $m\geqslant 1$. For $n\geqslant 0$, define

$$\sigma_{ns}=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}\sigma_{rk}\sigma_{n-r,s-k-1}\qquad (s\geqslant 1)$$

with $\sigma_{n0}=\Sigma_i\, i^n a_{i0}$. Now, for fixed n and s, assume inductively that

$$\sigma_{nj}=\sum_i i^n a_{ij}<\infty\qquad\text{for}\qquad j<s,\ n\leqslant m.$$

Then

$$\begin{aligned}\sigma_{ns}&=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}\sum_{m}^{\infty}m^r a_{mk}\sum_{l}^{\infty}l^{n-r}a_{l,s-k-1}\\&=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{i}^{\infty}\sum_{m+l=i}\left(\sum_{r=0}^{n}\binom{n}{r}m^r l^{n-r}\right)a_{mk}a_{l,s-k-1}\\&=\sum_i^{\infty} i^n\left(\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{m+l=i}a_{mk}a_{l,s-k-1}\right)\\&=\sum_i i^n a_{is}=L_{ns}.\end{aligned}$$

Therefore, $\sigma_{ns}=L_{ns}<\infty$ for $s\geqslant 0$, $n\leqslant m$.

Further, define

$$\phi_{ns}(z)=2^{-s}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\phi^{s+1})\qquad (n,\ s\geqslant 0)$$

where $\mathrm{Re}(z)\leqslant 0$. It is easy to verify directly that ϕ_{ns} satisfies the equation

$$\phi_{ns}(z)=\frac{1}{2s}\sum_{k=0}^{s-1}\sum_{r=0}^{n}\binom{n}{r}\phi_{r,k}(z)\phi_{n-r,s-k-1}(z)\qquad (s\geqslant 1).\tag{6}$$

Since $\phi_{n0}(z)=\mathrm{d}^n\phi/\mathrm{d}z^n$, we have $\phi_{n0}(0)=\sigma_{n0}$ by Dini's theorem (cf proof of theorem 1). Setting $z=0$ in (6), induction on s gives immediately that $\phi_{ns}(0)=\sigma_{ns}$, and hence

$$L_{ns}=2^{-s}\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\phi^{s+1})|_{z=0}\qquad (s\geqslant 0,\ n\leqslant m).$$

By theorem 3(i),

$$\begin{aligned}S_n(u)&=\tfrac{1}{2}Mu^2\sum_{s=0}^{\infty}2^{-s}(u_0-u)^s\frac{\mathrm{d}^n}{\mathrm{d}z^n}(\phi^{s+1})|_{z=0}\\&=\tfrac{1}{2}Mu^2\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left[\sum_s\left(\frac{u_0-u}{2}\right)^s\phi^{s+1}\right]_{z=0}.\end{aligned}$$

Since $|\phi(z)| \leqslant \phi(0) = 2u_0^{-1}$ for $\mathrm{Re}(z) \leqslant 0$, the geometric series may be summed to give the stated result for $S_n(u)$. Since $\phi(0) = 2u_0^{-1}$ and $\phi'(0) = 2u_0^{-2}$, we find that $S_0 = Mu$ and $S_1 = M$, and hence the X_i are the solution of (1.1) by theorem 1. If there exists m such that $S_n(u_0) = \infty$ for $n > m$, then since the first term in S_n involves L_{n0} $(=\sigma_{n0})$, we have $S_n = \infty$ $(n > m)$ for all t.

Remark. When appropriate, we may calculate the form of the higher moments from the above formula, e.g. for $m > 1$,

$$S_2(u) = 2M(u^{-1} - u_0^{-1}) + \tfrac{1}{2}Mu_0^2\phi''(0) = S_2(u_0) + M^2t.$$

We note that S_n is a polynomial in t of degree $n-1$ (cf Drake 1972).

4. The addition kernel

We now turn to the addition kernel $A = C = 0$, $B = 1$, and first dispose of the case where $\psi(z) = \Sigma\, a_{i0}z^i$ has radius of convergence $R > 1$. The result for $R = 1$ is subsequently obtained from this case by a limiting argument. To begin with we have the following.

Proposition 1. Let $A = C = 0$, $B = 1$, and assume that $\psi(z) = \Sigma a_{i0}z^i$ has radius of convergence greater than 1. Then the moments $S_n(t)$ of the X_i are finite for all $n \geqslant 0$ and all $t \geqslant 0$. In particular, $\pi = P$, $\mu = M$ and $X_i = c_i$ for all t.

Note that the finiteness of the moments at $t = 0$ is implied by the condition on ψ, since in general $S_n(0)$ can be expressed in terms of $\psi(1), \psi'(1), \ldots, \psi^{(n)}(1)$.

Proof. Let $v = u_0 - u$. Now $q(u) = 2u$, and hence for $v > 0$ theorem 3 implies that $L_{n+1,s} = \mathrm{d}L_{n,s}/\mathrm{d}u$. Hence

$$L_{0,s} = \frac{1}{s}\sum_{k=0}^{s-1} L_{0,s-k-1}\,\mathrm{d}L_{0,k}/\mathrm{d}u. \qquad (7)$$

Let $w(z) = z/\phi(z - v)$. Since $\phi(-v) \neq 0$ and $w'(0) = 1/\phi(-v) \neq 0$, w is analytic and univalent in a neighbourhood of $z = 0$. Thus we can write $z = wg(w)$ where $g(w)$ is analytic in a neighbourhood of $w = 0$, say $g(w) = \Sigma_{s=0}^{\infty} k_s(u)w^s$. Clearly g satisfies $g = \phi(wg - v)$, from which it is easy to verify, by implicit differentiation, that $g_w = gg_u$. Substituting the series expansion of g we find that $k_s = s^{-1}\,\Sigma_{n=0}^{s-1} k_n\,\mathrm{d}k_{s-n-1}/\mathrm{d}u$. Also $k_0 = g(0) = \phi(-v) = L_{0,0}$, so it follows from (7) and the above formula that $k_s = L_{0,s}$ for all $s \geqslant 0$. In other words $z = wg(w) = w\,\Sigma_{s=0}^{\infty} L_{0,s}w^s$.

Now fix $s \geqslant 0$ and let $f_s(\zeta) = (\phi(\zeta - v))^{s+1}$. Using $w = z/\phi(z - v)$ and Lagrange's expansion, we obtain

$$L_{0,s} = f_s^{(s)}(0)/(s+1)! \qquad \text{for all} \qquad s \geqslant 0 \qquad (8)$$

where differentiation is with respect to ζ. By assumption ψ has radius of convergence $R > 1$, so $\phi(\zeta - v)$ is analytic for all $\mathrm{Re}\,\zeta < v + \ln R$. Choose r such that $v < r < \min\{1, v + \ln R\}$. Then $f_s(\zeta - v)$ is analytic on $|\zeta| \leqslant r$, and for $|\zeta| = r$ we have $|f_s(\zeta)| = |\psi(\exp(\zeta - v))|^{s+1} \leqslant \psi(\exp(r - v))^{s+1} \leqslant \exp[(s+1)(r - v)]$. The Cauchy estimate on the circle $|\zeta| = r$, together with (8), imply that $L_{0,s} \leqslant \exp[(s+1)(r-v)]r^{-s}/(s+1)$, and so $\overline{\lim}_{s\to\infty} L_{0,s}^{1/s} \leqslant r^{-1}\exp(r - v)$. Thus $g(w) = \Sigma\, L_{0,s}w^s$ has radius of convergence at least

$r\exp(v-r)$, which is greater than v since $v<r\leqslant 1$. In particular $\pi=\Sigma_s L_{0,s}v^s=g(v)$ is finite.

To establish the finiteness of the higher moments, consider the relations $g_0=g$ and $dg_n/dw=\Sigma_{r=0}^{n}\binom{n}{r}g_{r+1}g_{n-r}$ for $n\geqslant 0$. This equation defines $g_{n+1}=g_0^{-1}[dg_n/dw-\Sigma_{r=0}^{n-1}\binom{n}{r}g_{r+1}g_{n-r}]$. Now g is zero-free in its region of analyticity, as follows from the equation $g(w)=\phi(wg(w)-v)$, and so all the g_n are analytic in the same region. It is now easy to see that $g_n(w)=\Sigma_{s=0}^{\infty}L_{n,s}w^s$ (using theorem 3(ii)), and so $S_n(u)=g_n(v)$ is finite for all v (the case $v=0$ being implied by hypothesis).

Finally $\pi=P$ and $\mu=M$ follow from theorem 1 in view of the finiteness of S_2.

To deal with the case where ψ has radius of convergence equal to 1 we need the following simple result.

Lemma. Let $f_i(\lambda)$, $i=1,2,\ldots$, be non-negative, increasing, continuous functions of λ for $\lambda\in[0,1]$, and set $F(\lambda)=\Sigma_{i=1}^{\infty}f_i(\lambda)$ (assumed finite for $\lambda<1$). Then

$$\sum_{i=1}^{\infty}f_i(1)=\lim_{\lambda\to 1}F(\lambda).$$

This is interpreted as meaning that one side is finite if and only if the other side is finite, and then the two are equal.

Proof. Let $\sigma_1=\Sigma_{i=1}^{\infty}f_i(1)\leqslant\infty$, $\sigma_2=\lim_{\lambda\to 1}F(\lambda)\leqslant\infty$. The assumptions imply that $\Sigma_{i=1}^{n}f_i(\lambda)\leqslant\Sigma_{i=1}^{n}f_i(1)\leqslant\sigma_1$. Letting $n\to\infty$ and then $\lambda\to 1$ we obtain $\sigma_2\leqslant\sigma_1$. Similarly $\Sigma_{i=1}^{n}f_i(\lambda)\leqslant F(\lambda)\leqslant\sigma_2$, so letting $\lambda\to 1$ and then $n\to\infty$ we get $\sigma_1\leqslant\sigma_2$.

We need to introduce some notation. Let the initial conditions $\{c_i(0)\}$ be such that $\psi(z)=(Mu_0)^{-1}\Sigma_i c_i(0)z^i$ has radius of convergence 1. For fixed $\lambda\in(0,1)$ define the new initial conditions

$$c_i^{\lambda}(0)=\lambda^i c_i(0).$$

(We denote the quantities defined in terms of the new initial conditions by a superscript, e.g. $P_0^{\lambda}=\Sigma_i\lambda^i c_i(0)$, though for convenience we omit the superscript in the limit $\lambda=1$.) Now set

$$b(\lambda)=\left(\sum_i\lambda^i a_{i0}\right)^{-1}=P_0/P_0^{\lambda}.$$

It follows that $b(\lambda)\lambda^i a_{i0}=P_0\lambda^i c_i(0)/P_0P_0^{\lambda}=c_i^{\lambda}(0)/P_0^{\lambda}=a_{i0}^{\lambda}$. A simple induction on j using (4) shows that $a_{ij}^{\lambda}=b(\lambda)^{j+1}\lambda^i a_{ij}$ for all $i,j\geqslant 0$. Writing $v(\lambda)=u_0^{\lambda}-u(\lambda)$ and noting that $\alpha^{\lambda}(u(\lambda))=\exp(u_0^{\lambda}-u(\lambda))=\exp(v(\lambda))$ (cf the appendix) we obtain

$$L_{n,s}^{\lambda}(u(\lambda))=\sum_{i=1}^{\infty}i^n b(\lambda)^{s+1}\lambda^i a_{is}\exp(-iv(\lambda)). \tag{9}$$

We can now prove the following.

Proposition 2. Let $u\in(0,u_0)$ be fixed. Then there exists a function u of λ such that $u(\lambda)\to u$ as $\lambda\to 1$, and $S_n(u)=\lim_{\lambda\to 1}S_n^{\lambda}(u(\lambda))$.

Proof. We can write $b(\lambda)=1/\psi(\lambda)$, where $\psi(z)=\Sigma_i a_{i0}z^i$. Substituting this into (9) and using theorem 3(i) we obtain

$$S_n^\lambda(u(\lambda))=M^\lambda u(\lambda)\psi(\lambda)^{-1}\sum_{i=1}^{\infty}\sum_{s=0}^{\infty} i^n a_{is}[\lambda\ \exp(-v(\lambda))]^i(v(\lambda)\psi(\lambda)^{-1})^s. \tag{10}$$

Set $v=u_0-u>0$. Since $u_0^\lambda\to u_0$ as $\lambda\to 1$, we have $v<u_0^\lambda$ for all λ sufficiently close to 1. Let $v(\lambda)=v+\ln\lambda<v<u_0^\lambda$. Then $\lambda\exp(-v(\lambda))=e^{-v}$ is independent of λ. Moreover $v(\lambda)>0$ for all λ sufficiently close to 1, and so (10) reduces to

$$S_n^\lambda(u(\lambda))=M^\lambda u(\lambda)\psi(\lambda)^{-1}\sum_{i=1}^{\infty}\sum_{s=0}^{\infty} i^n a_{is}\ e^{-iv}[(v+\ln\lambda)\psi(\lambda)^{-1}]^s.$$

Using the fact that $u_0^\lambda=\psi(\lambda)/\lambda\psi'(\lambda)$ it follows that $(v+\ln\lambda)\psi(\lambda)^{-1}$ is an increasing function of λ, since its derivative

$$-\psi(\lambda)^{-2}\psi'(\lambda)(v+\ln\lambda)+\psi(\lambda)^{-1}\lambda^{-1}=\psi(\lambda)^{-2}\psi'(\lambda)[u_0^\lambda-(v+\ln\lambda)]$$

is positive. Applying the lemma we find that

$$\lim_{\lambda\to 1} S_n^\lambda(u(\lambda))=Mu\psi(1)^{-1}\sum_i\sum_s i^n a_{is}\ e^{-iv}(v\psi(1)^{-1})^s=S_n(u)$$

since $\psi(1)=1$. The proof is complete.

Corollary. If $A=C=0$, $B=1$, then $\pi=P$, $\mu=M$, and $c_i=X_i$ for all $t\geqslant 0$.

Proof. If the radius of convergence of ψ is greater than one then this is proposition 1. Also, when ψ has radius of convergence one we know that $S_0^\lambda(u(\lambda))=M^\lambda u(\lambda)$ and $S_1^\lambda(u(\lambda))=M^\lambda$, by the same result. Then, for example, $\mu=S_1(u)=\lim_{\lambda\to 1} S_1^\lambda(u(\lambda))=\lim_{\lambda\to 1} M^\lambda=M$, by proposition 2. The proof for π is similar.

Remark. It is easy to show (cf Drake 1972) that

$$S_2^\lambda(u(\lambda))=S_2^\lambda(u_0^\lambda)(u_0^\lambda/u(\lambda))^2.$$

Thus by proposition 2 we have

$$S_2(u)=\lim_{\lambda\to 1} S_2^\lambda(u(\lambda))=(u_0/u)^2 S_2(u_0).$$

In other words, $S_2(u)$ is finite if and only if it is finite at time $t=0$. Similar considerations apply to the higher moments.

Appendix

Let $\Delta=B^2-AC$. There are five separate cases:

(i) $C>0, A=B=0$
(ii) $B>0, A=0$
(iii) $\Delta=0, A>0$
(iv) $\Delta>0, A>0$
(v) $\Delta<0$.

The formulae for $u(t)$ and t_0 $(C>0)$ are as follows.

Case	$u(t)$	t_0
(i)	$u_0-\frac{1}{2}CMt$	$2u_0/CM$
(ii)	$(u_0+C/2B)\exp(-BMt)-C/2B$	$(BM)^{-1}\ln(1+2Bu_0/C)$
(iii)	$\dfrac{u_0+B/A}{1+(Au_0+B)Mt/2}-\dfrac{B}{A}$	$\dfrac{2u_0}{M(Bu_0+C)}$
(iv)	$\dfrac{\surd\Delta}{A}\dfrac{\lambda\exp(Mt\surd\Delta)+1}{\lambda\exp(Mt\surd\Delta)-1}-\dfrac{B}{A}$ where $\lambda=\dfrac{Au_0+B+\surd\Delta}{Au_0+B-\surd\Delta}$	$\dfrac{1}{M\surd\Delta}\ln\left(\dfrac{(B+\surd\Delta)u_0+C}{(B-\surd\Delta)u_0+C}\right)$
(v)	$\dfrac{\surd-\Delta}{A}\tan(\theta-\frac{1}{2}(\surd-\Delta)Mt)-\dfrac{B}{A}$ where $\theta=\tan^{-1}\left(\dfrac{Au_0+B}{\surd-\Delta}\right)$	$\dfrac{2}{M\surd-\Delta}\tan^{-1}\left(\dfrac{\surd-\Delta\, u_0}{Bu_0+C}\right)$

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